Geometry and Mechanics

3 Part course for Columbia Splash 2017

Ben Israeli
byi2000@columbia.edu

March 25,2017

1 Calculus and Newtonian Mechanics

1.1 Kinematics and Calculus

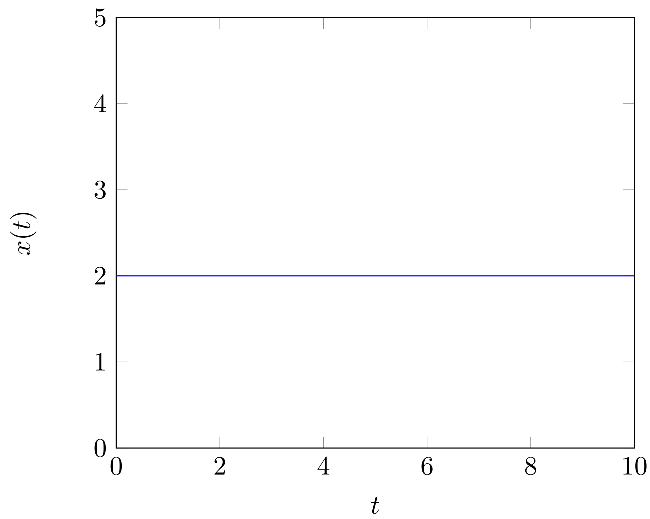
First, let's define the quantities we currently care about:

Definition 1.1. Position The location of an object in space.

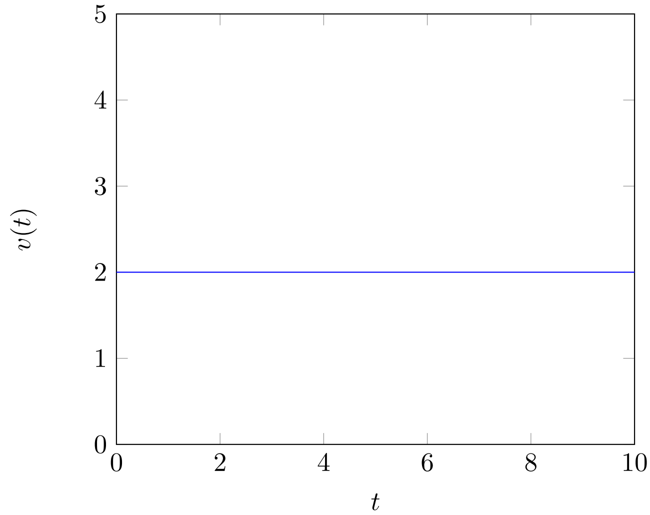
Definition 1.2. Velocity The rate at which an object's location changes.

Definition 1.3. Acceleration The rate at which an object's velocity changes.

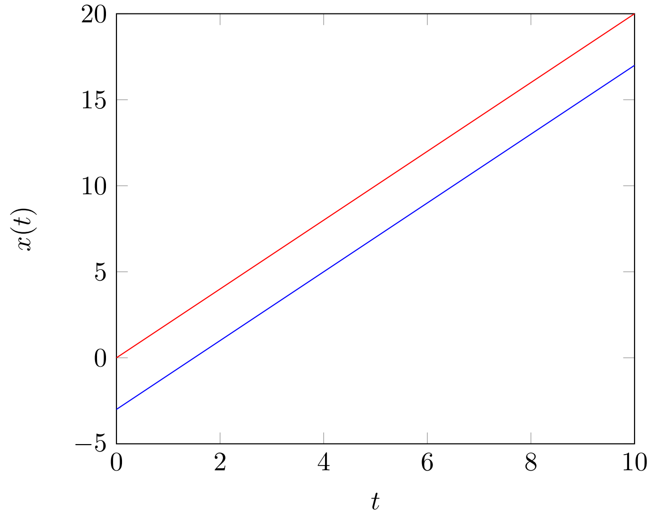
Note the recursive way these can be defined. Also, keep in mind that these are *vector* quantities. We will denote the position of an object as (x, y, z), the velocity as (v_x, v_y, v_z), etc.
Let's do some graphing to get a handle on how these quantities are related. We will stick with a one dimensional system with x, v, and a to keep things simple. Imagine an object sitting at rest at $x = 2m$. A plot of it's position as a function of time is shown below. Note that $v = 0m/s$ and $a = 0m/s^2$.



Now consider an object moving at a constant speed of 2m/s. A plot of it's velocity is shown below.



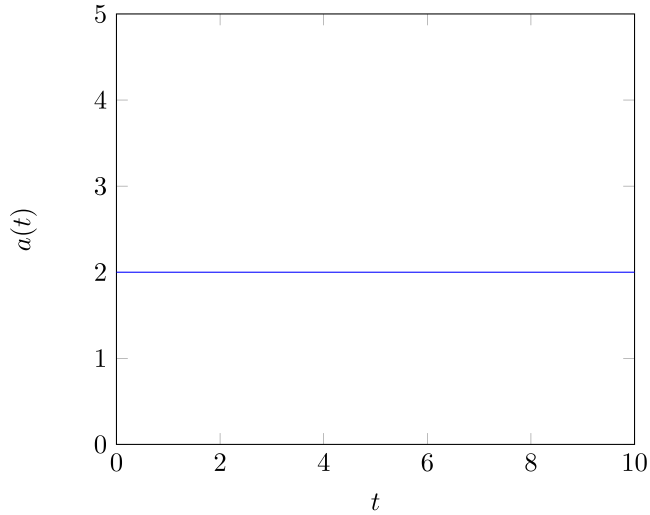
What does a plot of the object's position look like? It depends on where the object started. Here are two possible plots of the positions of objects with the above velocity.



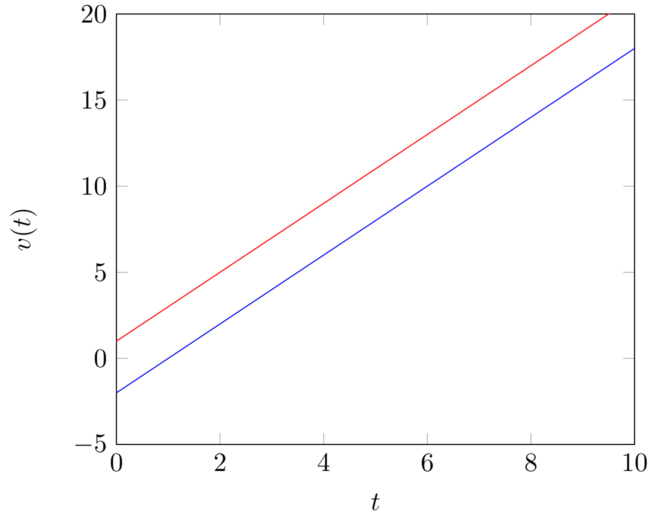
Notice that the velocity defined the *slope* of the position curve but not it's location. You can also see that the position plot describes the (signed) area under the velocity curve between two times. The red plot describes the area under the velocity curve between the origin and t, and the blue plot describes the area under the velocity curve between 2 and t (with sign determined by the direction relative to 2).

To summarize the velocity (the rate of change of position in time) describes the slope of position, and the position describes the cumulative effect of velocity over time. The units even cancel correctly. $m = m/s$ and $m/s = \frac{m}{s}$.

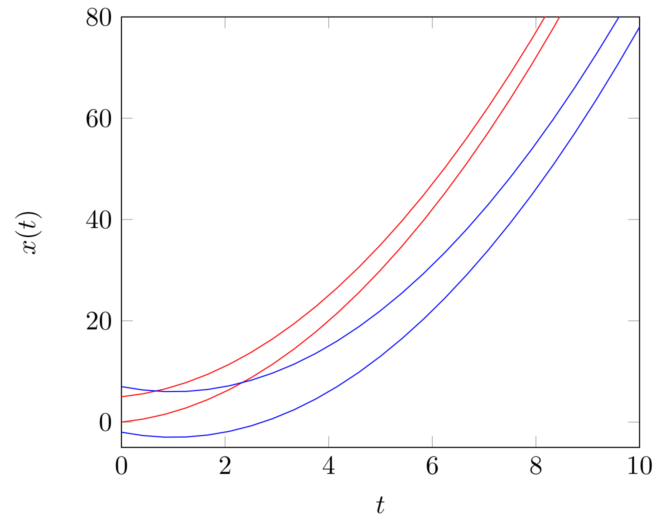
Now consider constant acceleration, say $a = 2m/s^2$.



Depending on the initial velocity, this can correspond to different velocity curves.



What about position? Working on the same intuition that position is the "area" under velocity, we can get several plots for each of the velocity curves. I chose two for each that fit nicely on the same graph.



You can see that the slope of the position curve at each point in time is the value of the velocity at that point in time.
We call this a derivative with respect to time. All that means is that the velocity is the rate of change of position over time.

Definition 1.4. Derivative A derivative with respect to some variable x of some function $f(x)$ is a function $f'(x)$ which is equal to the rate of change of f at x. On a graph, $f'(x)$ is the slope of $f(x)$. $f'(x)$ may also be denoted $\frac{df}{dx}$.

Notice the fraction notation is reminiscent of the formula for finding the slope of a line $m = \frac{\Delta y}{\Delta x}$. Indeed, if you zoom in on a regular (this is actually a term with specific meaning to mathematicians) spot on a graph of a function, it will look like a line. The derivative is essentially looking at the change after a very small displacement.
Note that we can have "higher order derivates": derivatives of derivatives. We write this like $\frac{d}{dx}\frac{d}{dx}f(x) = \frac{d^2}{dx^2}f(x) = f''(x)$.
There is also a mathematical term for the notion of "area". We call this an integral.

Definition 1.5. Definite Integral A function $F(x')$ is the integral of $f(x)$ between $x = a$ and $x = x'$ if the area under a graph of $f(x)$ versus x between $x = a$ and $x = x'$. This is denoted $\int_a^{x'} f(x)dx = F(x')$

Definition 1.6. Indefinite Integral If bounds are not specified, the integral of a function $f(x)$ is a family of functions equivalent up to an additive constant. $\int f(x)dx = F(x) + c$ with free variable c taking on any value and $\int_a^{x'} f(x)dx = F(x')$ for some a.

We can now give a rigorous description of the interplay between slope and area.

Theorem 1.1. *Fundamental Theorem of Calculus* $\int_a^x f'(x')dx' = f(x) + c$

Corollary 1.1.1. $\int_a^b f'(x) = f(b) - f(a)$

1.2 Properties of Integrals and Derivatives

Now we can start showing how to do arithmetic and algebra with these new tools. The following properties can be defined rigorously, but we are going to gloss over such details.
First not the following self-evident properties.

$$\frac{d}{dx}(f(x)+g(x)) = f'(x)+g'(x) \tag{1}$$

$$\int_a^b f(x)dx + \int_b^c f(x)dx = \int_a^c f(x)dx \tag{2}$$

1.2.1 Derivatives

Theorem 1.2. *Product Rule For $h(x) = f(x)g(x)$, $h'(x) = f'(x)g(x)+f(x)g'(x)$.*

Corollary 1.2.1. $\frac{d}{dx}ax^n = anx^{n-1}$

It may not be immediately intuitive that this is the case, but it seems reasonable enough that I will not be concerned with how this can be proved. From this (and the chain rule defined below), you can derive an equivalent quotient rule.
Also, by the way, $\frac{d}{dx}\sin(x) = \cos(x)$ and $\frac{d}{dx}\cos(x) = -\sin(x)$, which can be seen by visualizing their graphs.
Okay, so we can take derivatives of functions like $\sin(x)x^2$, but what about things like $\sin(x^2)$? These functions can be written as $f(g(x))$, where $f(x) = \sin(x)$ and $g(x) = x^2$ in this case. Just from the arithmetic of fractions, one might naively guess that $\frac{dx}{dy}\frac{dy}{dz} = \frac{dx}{dz}$ for some functions $y(z)$ and $x(y(z))$. This is actually correct, which we will not try to actually carefully derive. This is the chain rule.

Theorem 1.3. *Chain Rule For $h(x) = f(g(x))$, $h'(x) = g'(x)f'(g(x))$.*

For our example $\sin(x^2)$, we get $2x\cos(x^2)$

1.2.2 Integrals

There are 1001 fancy integration techniques that are of use in various circumstances. We will not worry much about them, but we will need to learn an important one, integration by parts. It allows us to integrate the product of two functions where we know the derivative of one and the integral of the other.
We are trying to solve $\int_a^b f(x)g(x)dx$ where we know $g(x) = h'(x)$ and we know $f'(x)$.
Recall the product rule: $\frac{d}{dx}(f(x)h(x)) = f'(x)h(x) + f(x)h'(x)$

$$
\begin{aligned}
\int_a^b f(x)g(x)dx &= \int_a^b (\frac{d}{dx}(f(x)h(x)) - f'(x)h(x))dx \\
&= \int_a^b (\frac{d}{dx} f(x)h(x))dx - \int_a^b f'(x)h(x)dx \\
&= f(b)h(b) - f(a)h(a) - \int_a^b f'(x)h(x)dx
\end{aligned}
$$

Introducing bar notation for differences

$$
\boxed{\int_a^b f(x)h'(x)dx = f(x)h(x)\Big|_a^b - \int_a^b f'(x)h(x)dx}
$$

1.3 Laws of Mechanics

1.3.1 Newton's Laws

1. Inertia: An object in motion stays in motion unless acted on by a force.
2. Forces: The acceleration of an object is in the direction of and proportional to the sum of the forces acting upon it.
3. Equal and opposite action: The force of one object on another is equal and opposite the force of the latter on the former.

$$
F = ma \tag{3}
$$

1.3.2 Conservation of Energy

Theorem 1.4. *Energy Conservation The total energy in an isolated system is constant.*

Definition 1.7. Kinetic Energy A body of mass m and velocity v possesses energy due to its motion equal to $\frac{1}{2}mv^2$.

Definition 1.8. Potential Energy A mechanical system may contain energy stored in forms other than kinetic energy, usually due to the relative position of objects. This is called potential energy.

Examples of potential energy include gravitational potential (mgh) and the energy stored in a spring ($\frac{1}{2}kx^2$ where x is the displacement of the spring from equilibrium and k is a constant).

1.4 Conservation of Momentum

Definition 1.9. Momentum The vector quantity momentum associated with an object is $p = mv$.

Theorem 1.5. *Momentum Conservation The total momentum of an isolated system is constant. This is true for each directional component.*

1.5 Worked Examples and Force Diagrams

I have several examples I aim to cover to demonstrate these concepts and the tools used to solve them. I have the problems and techniques memorized, and they are hard to typeset, so I will skip writing them in. You can find these problems in any intro mechanics textbook. The ones I would like to do are elastic collisions in 1 dimension, a mass on a ramp with friction, projectile motion, simple harmonic oscillation of a spring/mass system.
If I have extra time I may do a problem on masses and pulleys or talk about rotation.

2 Lagrangian and Hamiltonian Mechanics

2.1 Multivariable Calculus

Before we start, we need to get a handle functions of several variables and how to do basic calculus on these functions. So far we have used functions of one scalar variable, but we can have functions of vectors or functions of several scalars as well. These are denoted $f(x, y, z...)$.

2.1.1 Partial Derivatives

Derivatives work essentially the same way for such functions. For each variable we have a *partial derivative* denoted like $\frac{\partial}{\partial x} f(x, y)$. This is just the expression we get when we treat all variables but one as constants and take a derivative with respect to the remaining variable. For example, $\frac{\partial}{\partial x}(x^2 + xy + x^3 zy^2 + y + 4) = 2x + y + 3x^2 zy^2$. Notation such as $\left.\frac{\partial f(x,y)}{\partial x}\right|_{(1,3)}$ is used to denote, in this example, the value of this partial derivative at $(x, y) = (1, 3)$. An important property of partial derivatives is that $\frac{\partial}{\partial x}\frac{\partial}{\partial y} f(x, y) = \frac{\partial}{\partial y}\frac{\partial}{\partial x} f(x, y) = \frac{\partial^2}{\partial x \partial y} f(x, y)$.

2.1.2 Parameterization

Let's say we have a function $f(x, y)$, where x and y are variables that evolve in time. $f(x(t), y(t)$ will therefore evolve in time. In this way, a function of several variables can be treated as a function of one variable if its arguments are all functions of one variable. For example, $f(x, y) = xy$ $x(t) = t^2$ $y(t) = \sin(t)$ yields $f(x(t), y(t)) = t^2 \sin(t)$.

2.1.3 Total Derivative

In this example, at what rate does f change in time? We can calculate from $f(x(t), y(t)) = t^2 \sin(t)$ that $f'(t) = 2t \sin(t) + t^2 \cos(t)$. More generally, we also have the formula:

$$\frac{d}{dt} f(x(t), y(t), ...) = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + ... \tag{4}$$

It is easy to check that this gives the same answer for our example. This is known as a total derivative.

2.1.4 Path Integrals

We can also take the derivative of a parameterized function. This looks like $\int_a^b f(x(t), y(t))dt$. This might show up if, for example, f was some function of an object's position in the xy plane, and we wanted to sum up that function along a path taken by the object in the xy plane.

2.2 Lagrangian Mechanics

2.2.1 The Principle of Least Action

There is one such path integral that will be very useful. It is called the action. First, note that the kinetic energy of a system is a function of the velocities of its individual parts, and the potential energy is a function of time and the positions of the individual parts (and technically sometimes the velocities). Therefore, the Lagrangian, as defined below, is a function of the positions and velocities of the parts of a system.

Definition 2.1. Lagrangian The Lagrangian of a system is the difference between the total kinetic and potential energies of the system. It is denoted L=T-U.

So for a system with one degree of freedom, the Lagrangian will look like $L(q, \dot{q}, t)$, where q is that one degree of freedom, and $\dot{q}$ is the rate of change of this variable, its "velocity". q can be the position of a particle, but it also can be any other variable that specifies the state of the system. This is what we mean by a degree of freedom. A system with several degrees of freedom would have a Lagrangian denoted $L(q_1, q_2..., \dot{q}_1, \dot{q}_2, ..., t)$ or more succinctly $L(\vec{q}, \dot{\vec{q}}, t)$, where the vectors are just lists of degrees of freedom. Now that we have that out of the way, we can define the action.

Definition 2.2. Action For a motion of a system specified by some path $\vec{q}(t)$, the action of the motion is the path integral of the Lagrangian along this motion. $S[q(t)] = \int_a^b L(\vec{q}(t), \dot{\vec{q}}(t), t)dt$.

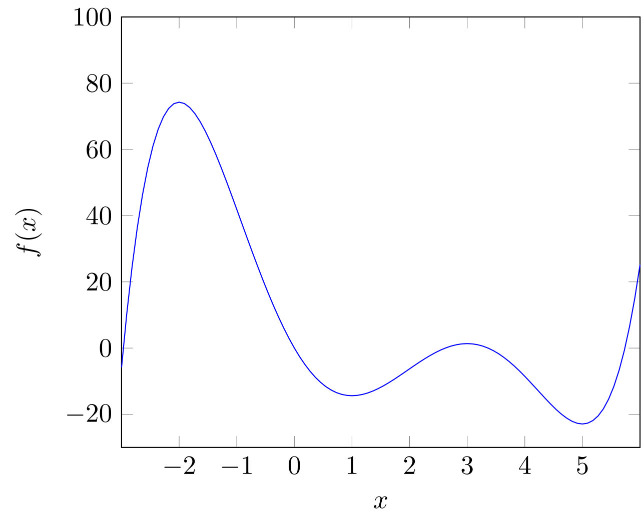
The action is what we call a functional, a function of functions. It take some function $q(t)$ and spits out a number. Ok now the good part.

Theorem 2.1. *Hamilton's Principle of Least Action The path taken by a system with Lagrangian $L(\vec{q}, \dot{\vec{q}}, t)$ will be some $\vec{q}(t)$ such that the action $S[q(t)] = \int_a^b L(\vec{q}(t), \dot{\vec{q}}(t), t)dt$ is minimized.*

This may seem somewhat arbitrary right now, but we will see shortly that this is an elegant and powerful tool for understanding mechanics problems.

2.2.2 Aside: Extrema of Functions

Before we continue, let's get a handle on what we mean by minimization. For simplicity, we will only care about continuous smooth functions. Consider the function plotted below.



We can see that is has a maximum at $x = 5$ and a minimum at $x = -2$. If we only consider the domain $(-3, 6)$, we can say that it has a *global* maximum at $x = 5$ and *global* minimum at $x = -2$. Now consider $x = 1$. There is a region around this point where $f(x) \geq f(1)$ for any x in the region. We can therefore call $x = 1$ a *local* minimum. Similarly, $x = 3$ is a *local* maximum.
Notice that, at any extremum (maximum or minimum), the slope of the function is zero. In other words, $f'(x) = 0$. This is an important property of continuous functions; their derivatives are zero at extrema.

2.2.3 The Euler-Lagrange Equations

We now have a useful property for the motion of a system. The derivative of the action will be zero for the path taken by the system. But wait, the action is a functional, taking functions as its input. What can we possibly mean by a derivative of a functional. It turns out functional derivatives can be well defined, which is a topic in the "calculus of variations", a slippery topic we will not delve too far into.
However, we can make an intuitive argument that will get us far enough. Let some path $q(t)$ (dropping vector notation for my own sanity) minimize the action of our system for some motion from $q(0) = 0$ to $q(a) = b$. Now, take some arbitrary continuous function $f(t)$ such that $f(0) = f(a) = 0$. Consider the path $\tilde{q}(t) = q(t) + \alpha f(t)$. Since f is conveniently zero at the endpoints, picking different values of f can give us any path from $(0, 0)$ to (a, b). For any f, picking an arbitrarily small α will make $\tilde{q}(t)$ arbitrarily close to $q(t)$. (I seriously hope I try to draw this.)
Now, consider $S[\tilde{q}(\alpha, t)]$ as being a function of α. $S[q(t)]$ takes a path and gives a number, and $\tilde{q}(\alpha, t)$ can be thought of as taking a number α and spitting out a path. So, we can think of $S[\tilde{q}(\alpha, t)]$ as taking a number α and spitting out a number.We can then even take the derivative of $S[\tilde{q}(\alpha, t)]$ with respect to α.
If $q(t)$ locally minimizes the action, it must be the case that $S[\tilde{q}(\alpha, t)] \geq S[q(t)]$ for small values of α. In other words, $S[\tilde{q}(\alpha, t)]$ has a local minimum at $\alpha = 0$, and therefore $\left.\frac{dS[\tilde{q}(\alpha,t)]}{d\alpha}\right|_{\alpha=0} = 0$.

Let's now expand that last expression (dropping the tilde).

$$0 = \left.\frac{dS[\tilde{q}(\alpha,t)]}{d\alpha}\right|_{\alpha=0} = \frac{d}{d\alpha}\int_0^a L(q(t,\alpha),\dot{q}(t,\alpha),t)dt$$

We can push the derivative inside the integral since the limits of the integral are fixed, and the integral is not over α.

$$= \int_0^a \frac{d}{d\alpha} L(q(t,\alpha),\dot{q}(t,\alpha),t)dt$$

Expanding this as a total derivative, noting t is independent of α

$$= \int_0^a \left(\frac{\partial L}{\partial q}\frac{\partial q}{\partial \alpha} + \frac{\partial L}{\partial \dot{q}}\frac{\partial \dot{q}}{\partial \alpha}\right)dt$$
$$= \int_0^a \left(\frac{\partial L}{\partial q}f(t) + \frac{\partial L}{\partial \dot{q}}\frac{df}{dt}\right)dt$$

We can integrate the second term by parts.

$$= \left.\frac{\partial L}{\partial \dot{q}}f(t)\right|_0^a + \int_0^a \left(\frac{\partial L}{\partial q}f(t) - \left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}}\right)f(t)\right)dt$$

The first term drops since $f(0) = f(a) = 0$.

$$= \int_0^a \left(\frac{\partial L}{\partial q}f(t) - \left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}}\right)f(t)\right)dt$$
$$0 = \int_0^a \left(\frac{\partial L}{\partial q} - \left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}}\right)\right)f(t)dt$$

This only holds for arbitrary $f(t)$ if:

$$\boxed{\frac{\partial L}{\partial q_i} - \left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i}\right) = 0} \tag{5}$$

The i subscripts mean that this equation holds separately for each degree of freedom of the system, indexed by i. If you look carefully, the previous derivation works whether q and $\frac{q}{dq}$ are scalars or vectors.
These are the **Euler-Lagrange Equations**. For each degree of freedom of a system, they give a differential equation which specifies the motion of that degree of freedom. This is a very powerful and flexible tool. Given just the Lagrangian of a system (defined in whatever coordinates are most convenient), we can fully describe its behavior. Hopefully this will become clear in the following examples.

2.2.4 Example Problems

Once again, I have problems memorized which would be quite arduous to write out here. In class I will probably do a hoop on a ramp and maybe two couple oscillators.

2.3 Hamiltonian Mechanics

2.3.1 Conjugate Momenta and the Hamiltonian

Notice that kinetic energy terms generally look something like $\frac{1}{2}m\dot{q}^2$. For example, translational and rotational kinetic energy: $\frac{1}{2}mv^2$, $\frac{1}{2}I\omega^2$. Also, in most systems anyway, the potential energy is not dependent on velocities. So, usually, for some Lagrangian with a degree of freedom q_j, $\frac{\partial L}{\partial \dot{q}_j} = m\dot{q}$. If q were a cartesian position of a particle, then this would be the momentum of the particle along this coordinate. We can define a generalized momentum corresponding to some generalized position q_j to be

$$p_j = \frac{\partial L}{\partial \dot{q}_j} \tag{6}$$

In the case of a position variable that is actually a cartesian position of a particle, the *conjugate* momentum is the actual momentum of that particle along that coordinate. In the case of a variable describing the angular orientation of an object about some axis, the *conjugate* momentum is the angular momentum of that object along that axis. Substituting into the Euler-Lagrange Equation gives us:

$$\dot{p}_j = \frac{\partial L}{\partial q_j} \tag{7}$$

Solving (6) allows us to think of our velocities as functions of positions, momenta, and time $\dot{q}_j(q_k, p_k, t)$. Then we can define the **Hamiltonian**, a function of positions, momenta, and time.

$$H(q_k, p_k, t) = \sum_j p_j \dot{q}_j(q_k, p_k, t) - L(q_k, \dot{q}_j(q_k, p_k, t), t) \tag{8}$$

Notice that $H = T + U$ for usual systems.

2.3.2 Hamilton's Equations

We are now set to derive a new set of equations equivalent to the Euler-Lagrange equations.

Taking the total time derivative of H:

$$\frac{dH}{dt} = \sum_k \left(\frac{\partial H}{\partial q_k}\frac{dq_k}{dt} + \frac{\partial H}{\partial p_k}\frac{dp_k}{dt}\right) + \frac{\partial H}{\partial t}$$

From (8), we can also derive:

$$\frac{dH}{dt} = \sum_k \left(\dot{q}_k \frac{dp_k}{dt} + p_k \frac{dq_k}{dt} - \frac{\partial L}{\partial q_k}\frac{dq_k}{dt} - \frac{\partial L}{\partial \dot{q}_k}\frac{d\dot{q}_k}{dt}\right) - \frac{\partial L}{\partial t}$$

Making substitutions in the third and fourth term then cancelling the second and fourth

$$\frac{dH}{dt} = \sum_k (\dot{q}_k \frac{dp_k}{dt} - \dot{p}_k \frac{dq_k}{dt}) - \frac{\partial L}{\partial t}$$

Equating like terms in the first and last expression gives us **Hamilton's equations**.

$$\boxed{\dot{q}_k = \frac{\partial H}{\partial p_k}; \quad \dot{p}_k = -\frac{\partial H}{\partial q_k}} \tag{9}$$

Substituting these into the final expression and cancelling gives us:

$$\frac{dH}{dt} = \frac{\partial H}{\partial t} \tag{10}$$

This tells us that if H does not explicitly dependent on t, then it is a conserved quantity. Noting that usually $H = T + U$, the Hamiltonian can be thought of as the total energy of a system, and we just derived conservation of energy starting only the Principle of Least Action.

2.3.3 Worked Examples

Now that we have yet another tool set, let's solve some problems. In class I will likely do coupled oscillators again, and maybe a spherical pendulum (nice conservation of angular momentum example).

3 Geometric Methods

I decided to write this by hand because I will have to draw a lot and typesetting takes forever.

Geometry and Mechanics Pt. 3

Configuration Space - space of all possible configurations of a system.

Examples:

Ball on table → 2D euclidean space - xy or Rθ coords

Rigid 1D pendulum → circle - 0=2π wraps back

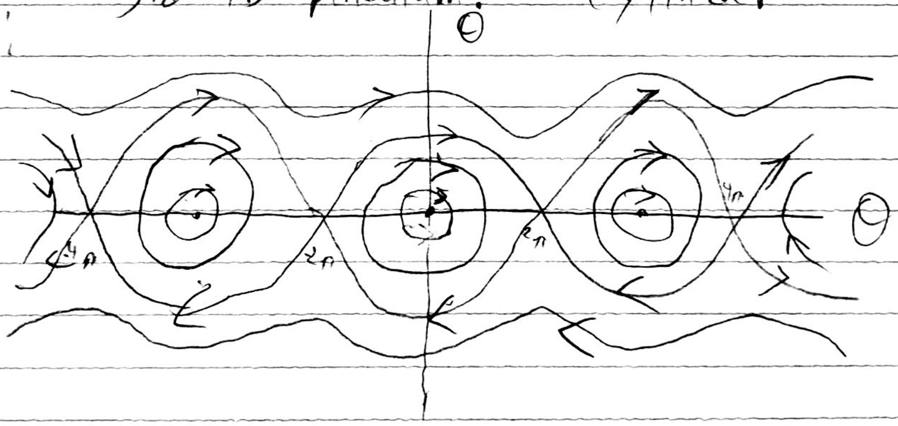
(should have props on hand)

Phase Space - space of all possible positions and velocities of the degrees of freedom of a system. Tangent space at each point in phase space.

Ball on table → 4D euclidean space

Rigid 1D pendulum → cylinder

Flow lines:



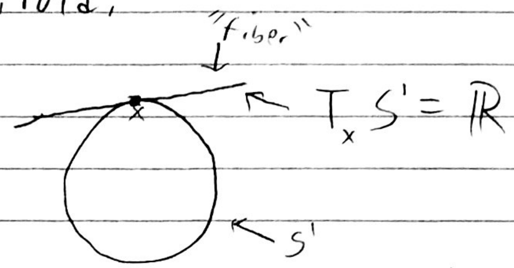
Some quick differential geometry

Manifold: any space which looks Euclidean (flat) when you zoom in on it.

Examples: $\mathbb{R}^n$, sphere, donut, spacetime...

Tangent space: Space of vectors tangent to a point on a manifold. This is $\mathbb{R}^n$ for an n dimensional manifold.

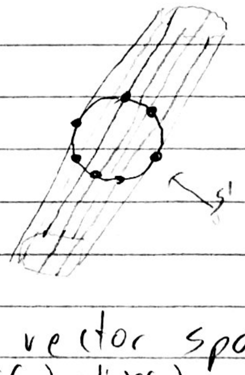
Example: circle S^1



("fiber bundle")

Tangent bundle: Manifold created by stitching together tangent spaces for every point on a manifold.

Example: TS^1 = infinite cylinder



A vector field is a section of a tangent bundle. It picks a value from the tangent space at each point. It is a map $X: M \mapsto TM$ s.t. $X(p) \in T_p M$

Dual space to a vector space: space of linear functions $(af(x) + bf(y) = f(ax+by))$ on the vector space

any $a \in V^*$ can be written

$$a(\vec{v}) = a(v_1, v_2, v_3 \ldots) = a_1 v_1 + a_2 v_2 + a_3 v_3 + \ldots$$

so a can be specified as $a(\vec{v}) = \vec{a} \cdot \vec{v}$

V^* is a vector space isomorphic to V

Dual Vectors are functions on vectors, and vectors are functions on dual vectors.

We can define a dual space to a tangent space T_pM.
This is the cotangent space T_p^*M.

We can also stitch together the cotangent spaces to get a cotangent bundle T^*M
(another fiber bundle)

And similarly, we can take sections of the cotangent bundle, maps $\gamma: M \to T^*M$ s.t. $\gamma(p) \in T_pM$.

So, a dual vector (field) takes a vector (field) and spits out a number (at each point on M).
A vector (field) takes a dual vector (field) and spits out a number (at each point on M).
In both cases, these maps behave linearly.
We can generalize this to objects that take several vectors and/or dual vectors and spit out a number.

Rank (r,s) Tensor - A function with r arguments which are vectors and s arguments which are dual vectors and which is linear in each of its arguments (multilinear).

Vectors are rank $(0,1)$, and dual vectors are rank $(1,0)$.

If you feed a tensor only some of its arguments, it becomes a lower rank tensor.
$$T(\cdot, \cdot, \cdot) \to T(v, \cdot, \cdot)$$

So, matrices are $(1,1)$ tensors.
If you feed a matrix a vector, it spits out another vector (a $(0,1)$ tensor).

k-form: an antisymmetric rank $(k,0)$ tensor
Antisymmetric means swapping any two arguments changes the result by only a sign.

$$T(v_1, v_2, v_3) = -T(v_2, v_1, v_3)$$

2-form: an antisymmetric rank $(2,0)$ tensor
multilinear object taking

2 vectors $\longrightarrow$ a number
1 vector $\longrightarrow$ a dual vector

differential 2-form: A two form defined at each point on a manifold.

2 vector field $\rightarrow$ real valued function on M
1 vector field $\rightarrow$ a dual vector field

You can write 2-forms as matrices

$$T(u, v) = (u_1, u_2, u_3, \ldots) \begin{pmatrix} & & \\ & T & \\ & & \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ \vdots \end{pmatrix}$$

symplectic form: closed nondegenerate 2-form
derivative 0 (don't worry about that too much)
nonzero at all points on M

Thm (Darboux): For a symplectic form w on a manifold M, coordinates can be found such that

$$w = \begin{pmatrix} 0 & \begin{matrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{matrix} \\ \begin{matrix} -1 & & 0 \\ & \ddots & \\ 0 & & -1 \end{matrix} & 0 \end{pmatrix} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

Symplectic manifold: A manifold on which a symplectic form can be defined.
symplectic: $\mathbb{R}^n$, torus (donut), infinite cylinder
nonsymplectic: sphere, mobius strip

All symplectic manifolds are even dimensional.
Notice that the previously defined w requires this.

Thm: Any cotangent bundle on a manifold is a symplectic manifold.

The proof takes too long to fit in 1 hr, but it essentially comes from the fact that a cotangent bundle has two sets of coordinates (one on the base manifold and one on the cotangent space) which have an inherently complementary structure.

So on any phase space with coords (q_i, p_i), there is a 2-form $w(u, v)$ s.t.

$$w(\cdot, \cdot) = \begin{pmatrix} 0 & \begin{matrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{matrix} \\ \begin{matrix} -1 & & 0 \\ & \ddots & \\ 0 & & -1 \end{matrix} & 0 \end{pmatrix}$$

Let's consider some phase space M with a Hamiltonian $H: M \to \mathbb{R}$

Consider $\vec{\nabla} H$, the gradient of H. This is a vector field such that each component is the derivative of H along that component's direction. The vector field points along the direction of steepest increase for H. $\nabla H = \left(\frac{\partial}{\partial q_1} H, \frac{\partial}{\partial q_2} H \ldots \frac{\partial}{\partial p_1} H, \frac{\partial}{\partial p_2} H \ldots \right)$

(We will justify this definition later)

Let's say we want to find a vector field X_H such that $\omega(Y, X_H) = Y \cdot \nabla H$ for any Y

let $X_H = (\alpha_1 \dots \alpha_n, \beta_1 \dots \beta_n)$ $Y = (a_1, \dots a_n, b_1, \dots b_n)$

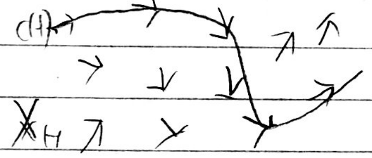
Hamiltonian vector field

$$(a_1 \dots a_n, b_1 \dots b_n)\begin{pmatrix} 0 & \begin{matrix} 1 & & \\ & \ddots & \\ & & 1\end{matrix} \\ \begin{matrix} -1 & & \\ & \ddots & \\ & & -1\end{matrix} & 0 \end{pmatrix}\begin{pmatrix}\alpha_1 \\ \vdots \\ \alpha_n \\ \beta_1 \\ \vdots \\ \beta_n\end{pmatrix} = \begin{pmatrix} a_1 \\ \vdots \\ a_n \\ b_1 \\ \vdots \\ b_n\end{pmatrix} \cdot \begin{pmatrix} \frac{\partial}{\partial q_1} H \\ \vdots \\ \frac{\partial}{\partial p_1} H \\ \vdots \end{pmatrix}$$

$$\alpha_1 b_1 + \alpha_2 b_2 + \dots - \beta_1 a_1 - \beta_2 a_2 - \dots = a_1 \frac{\partial}{\partial q_1} H + a_2 \frac{\partial}{\partial q_2} H + \dots + b_1 \frac{\partial}{\partial p_1} H + b_2 \frac{\partial}{\partial p_2} H$$

so $\alpha_i = \frac{\partial}{\partial p_i} H$ $\beta_i = -\frac{\partial}{\partial q_i} H \rightarrow X_H = \left(\frac{\partial}{\partial q_1} H, \dots \frac{\partial}{\partial q_n} H, \frac{\partial}{\partial p_1} H, \dots \frac{\partial}{\partial p_n} H\right)$

Now imagine some path $\vec{c}(t)$ that follows X_H.
It moves in the direction of X_H at the speed $|X_H|$
(Like the flow lines I drew earlier)



$\frac{d}{dt} \vec{c}(t) = X_H$

along $c(t)$, $\boxed{\dot{p}_i = -\frac{\partial H}{\partial q_i} \quad \dot{q}_i = \frac{\partial H}{\partial p_i}}$

Hamilton's equations!

We can see that $\nabla H \cdot X_H = \sum_i \frac{\partial}{\partial q_i} H \frac{\partial}{\partial p_i} H - \sum_i \frac{\partial}{\partial p_i} H \frac{\partial}{\partial q_i} H = 0$
the motion defined by X_H does not change the energy.
So we have a way of deriving Hamilton's equs and conservation of energy.

Lets say we have some function $f(q, p)$ on phase space
How does it vary in time?

$$\frac{d}{dt} f = \sum_i \frac{\partial f}{\partial q_i} \dot{q}_i + \sum_i \frac{\partial f}{\partial p_i} \dot{p}_i = \sum_i \left(\frac{\partial f}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial H}{\partial q_i}\right)$$

We define this as the Poisson Bracket $\frac{d}{dt} f = \{f, H\}$
Where for any to functions $f(q,p), g(q,p)$, $\{f, g\} = \sum_i \left(\frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i}\right)$

For a conserved quantity, $\frac{d}{dt} f = \{f, H\} = 0$

We also have a Lie Bracket of two vector fields.

$$[X,Y] = X\cdot\nabla Y - Y\cdot\nabla X = \begin{pmatrix} \vec{X}\cdot\vec{\nabla}Y_1 - Y\cdot\vec{\nabla}X_1 \\ \vec{X}\cdot\vec{\nabla}Y_2 - Y\cdot\vec{\nabla}X_2 \\ \vdots \\ \vdots \end{pmatrix}$$

$\mathcal{L}_X Y = [X,Y]$ is equivalent and called the Lie Derivative

It is the rate of change of Y along the flow of X.

It can be defined more generally on other objects. For example $\mathcal{L}_{X_H} W = 0$.

This means that the motion of a system with Hamiltonian H preserves the symplectic structure (the unique relation of p's to q's).

This is why we believe our definition of X_H

There is a beautiful property that

$$X_{\{f,g\}} = [X_f, X_g]$$

This can be proved by substituting in formulas.

Equivalently $\{f,g\} = \mathcal{L}_{X_f} f$

So if $\{f,g\}=0$ ("they Poisson commute")

A system with Hamiltonian f has g as a conserved quanti

A system with Hamiltonian g has f as a conserved quant

$$[X_f, X_g] = 0 \quad \text{("the vector fields commute")}$$

The flow of f Lie drags X_g

The flow of g Lie drags X_f

Now Lagrangians

$H : T^*M \to \mathbb{R}$ similarly $L : TM \to \mathbb{R}$

Fiber Derivative: Map from $TM \to T^*M$

$v_p, w_p \in T_pM$ for som $p \in M$

$\underbrace{(FL(v_p))}_{\in T^*M}(w_p) = \frac{d}{ds}\Big|_{s=0} L(v_p + s w_p)$

$\therefore FL(v_p) = \frac{\partial L(q,\dot q)}{\partial \dot q}\Big|_{q=p,\ \dot q = v}$

For some point p in the configuration space M, FL takes a tangent vector (velocity) at p and turns it into a linear function on the tangent space describing the rate of change of the Lagrangian L as you vary v.

Called a Fiber derivative since it acts within a fiber T_pM

$w_L = (FL)^* w$ "pullback" of w on T^*M to TM

$\therefore FL : TM \to T^*M$ also defines a map of tensors on T^*M to tensors on TM

In "wedge notation" which we will not try to use

$w = \sum_i dq_i \wedge dp_i \quad w_L = \sum_i dq_i \wedge d\frac{\partial L}{\partial \dot q_i}$ (change of variable as expected)

For some $L : TQ \to \mathbb{R}$ we can define an Action $A : TQ \to \mathbb{R}$

$A(v_x) = FL(v_x)\, v_x$ and energy $E = A - L$

tangent vector v at point x

Lagrangian vector field: X_E on TQ s.t. $w_L(Y, X_E) = Y \cdot DE$

$$w_L((e_1, e_2), (e_3, e_4)) = \frac{\partial}{\partial q}\left(\frac{\partial L}{\partial \dot q} \cdot e_3\right) \cdot e_1 - \frac{\partial}{\partial q}\left(\frac{\partial}{\partial \dot q} \cdot e_1\right) \cdot e_3$$
$$+ \frac{\partial}{\partial \dot q}\left(\frac{\partial L}{\partial \dot q} \cdot e_4\right) \cdot e_1 - \frac{\partial}{\partial \dot q}\left(\frac{\partial L}{\partial \dot q} \cdot e_2\right) \cdot e_3$$

Let $X_E(q,\dot q) = (Y(q,\dot q), Z(q,\dot q))$

$A(q,\dot q) = \frac{\partial}{\partial \dot q} L(q,\dot q)\cdot \dot q \qquad E(q,\dot q) = \frac{\partial}{\partial \dot q}L(q,\dot q)\cdot\dot q - L$

$$\nabla E(q,\dot q)\cdot(e_1,e_2) = \frac{\partial}{\partial q}E(q,\dot q)\cdot e_1 + \frac{\partial}{\partial \dot q}E(q,\dot q)\cdot e_2$$

$$= \frac{\partial}{\partial q}\left(\frac{\partial L}{\partial \dot q}\cdot\dot q - L\right)\cdot e_1 + \frac{\partial}{\partial \dot q}\left(\frac{\partial L}{\partial \dot q}\cdot \dot q - L\right)\cdot e_2$$

$$= \left(\underset{\text{matrix}}{\frac{\partial}{\partial q}\frac{\partial L}{\partial \dot q}}\cdot \dot q - \frac{\partial L}{\partial q}\right)\cdot e_1 + \left(\underset{\text{matrix}}{\frac{\partial}{\partial \dot q}\frac{\partial L}{\partial \dot q}}\dot q + \cancel{\frac{\partial L}{\partial \dot q}} - \cancel{\frac{\partial L}{\partial \dot q}}\right)\cdot e_2$$

$$w_L((e_1,e_2), X_E) = \nabla E\cdot(e_1,e_2)$$

$$\frac{\partial}{\partial q}\left(\frac{\partial L}{\partial \dot q}\cdot Y\right)\cdot e_1 - \frac{\partial}{\partial q}\left(\frac{\partial L}{\partial \dot q}\cdot e_1\right)\cdot Y + \frac{\partial}{\partial \dot q}\left(\frac{\partial L}{\partial \dot q}\cdot Z\right)\cdot e_1 - \frac{\partial}{\partial \dot q}\left(\frac{\partial L}{\partial \dot q}\cdot e_2\right)\cdot Y$$

$$= \left(\frac{\partial}{\partial q}\left(\frac{\partial L}{\partial \dot q}\cdot\dot q\right) - \frac{\partial L}{\partial q}\right)\cdot e_1 + \frac{\partial}{\partial \dot q}\left(\frac{\partial L}{\partial \dot q}\cdot \dot q\right)\cdot e_2$$

match like terms

$$\frac{\partial}{\partial \dot q}\left(\frac{\partial L}{\partial \dot q}\cdot e_2\right)\cdot Y = \frac{\partial}{\partial \dot q}\left(\frac{\partial L}{\partial \dot q}\cdot\dot q\right)\cdot e_2 \Rightarrow Y(q,\dot q) = \dot q \qquad \left(\frac{dq}{dt}=\dot q\right)$$

$$\cancel{\frac{\partial}{\partial q}\left(\frac{\partial L}{\partial \dot q}\cdot\dot q\right)\cdot e_1} - \cancel{\frac{\partial}{\partial q}\left(\frac{\partial L}{\partial \dot q}\cdot e_1\right)\cdot \dot q} + \frac{\partial}{\partial \dot q}\left(\frac{\partial L}{\partial \dot q}\cdot Z\right)\cdot e_1 = \left(\frac{\partial}{\partial q}\left(\frac{\partial L}{\partial\dot q}\cdot\dot q\right) - \frac{\partial L}{\partial q}\right)\cdot e_1$$

$$\frac{\partial}{\partial \dot q}\left(\frac{\partial L}{\partial \dot q}\cdot Z\right) = \frac{\partial}{\partial q}\left(\frac{\partial L}{\partial \dot q}\cdot \dot q\right) - \frac{\partial L}{\partial q}$$

Along some path $q(t)$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot q}\right) = \frac{\partial}{\partial q}\left(\frac{\partial L}{\partial \dot q}\right)\cdot\frac{dq}{dt} + \frac{\partial}{\partial \dot q}\left(\frac{\partial L}{\partial \dot q}\right)\cdot \frac{d\dot q}{dt}$$

The flow along X_E is defined by

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot q}\right) = \frac{\partial L}{\partial q}$$ (sign errors somewhere above)

Euler-Lagrange!